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ASHIS SEN GUPTA

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TESTS FOR STANDARDIZED GENERALIZED VARIANCES OF MULTIVARIATE NORMAL POPULATIONS OF POSSIBLY DIFFERENT DIMENSIONS[1]

Ashis Sen Gupta[2]
Stanford University

1. Introduction.

Let X be a p-dimensional random vector variable with dispersion matrix Σ. Two well-known measures of multidimensional scatter, obtained by generalizing the variance, the univariate measure, are Σ and the generalized variance (GV), $|\Sigma| = \det(\Sigma)$, introduced by Wilks (1932, 1967). For multivariate normal populations, Likelihood Ratio Tests (LRT's) for Σ's, of course of same dimensionalities, and some optimum properties of these tests are known. But, when multidimensional scatter of populations of different dimensions need to be compared, these tests cannot be defined. However, using $|\Sigma|^{1/p}$, which we will nomenclature as Standardized Generalized Variance (SGV), such comparisons become meaningful. Since $|\Sigma|$ represents the volume in p-dimensions, note that $|\Sigma|^{1/p}$ becomes a measure so scaled as to become comparable with scatter for a scalar random variable. Apart from this generality, need for tests of generalized variances has been also felt, on its own right. $|\Sigma|$, being a scalar, is more suitable and easier to work with than the matrix Σ. Hoel (1937) was

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[2] Part of this paper was presented by the author, as an invited speaker, at the Conference on Applications of Numerical Analysis and Special Functions in Statistics, University of Maryland, October 6, 1980.

probably first to realize this need and later Eaton (1967) studied some problems of statistical inference associated with a single GV. Bickel (1965) and others have used "generalized variance efficiency" for comparing various test procedures. The GV has been extensively used in applied research, e.g., by Goodman (1966) in Agricultural Statistics, Gnanadesikan and Gupta (1970) in Ranking and Selection, Arvanitis and Afonja (1971) in Sample Survey, Kiefer and Studden (1976) in The Theory of Optimal Designs, etc. While the estimation, e.g., van der Vaart (1965), Shorrock and Zidek (1976), and the distribution, e.g., Mathai (1972) have been studied in some detail, little seems to be known about tests for GV's. This paper attempts to bridge that gap.

Suppose $\underset{\sim}{X}_i$'s are independently distributed as $N_{p_i}(\mu_i, \Sigma_i)$, Σ_i being general dispersion matrices, $i = 1,\ldots,k$. LRT's are derived for $H_{01} : |\Sigma_i|^{1/p_i} = \sigma_0^2$ (given) > 0, for some fixed i; $H_{02} : |\Sigma_i|^{1/p_i} = |\Sigma_j|^{1/p_j}$, for some fixed i and j and finally $H_{03} : |\Sigma_i|^{1/p_i}$ all equal, $i = 1,\ldots,k$ against appropriate two-sided alternatives. The test criteria turn out to be quite elegant multivariate analogues to those in the univariate cases.

The solutions to the distributional problems associated with the various test statistics considered above need extensive use of Special Functions. The exact distributions for both the null and nonnull cases are presented for most of the above test criteria. The percentage points of these distributions can be obtained from existing mathematical tables since the distributions are represented in suitable computable forms. Examples of construction of tables and the general procedure of obtaining them for such computable forms of the distributions exist in current literature, e.g., Mathai and Katiyar (1979). Further, many of the existing tables can also be exploited to give the percentage points. Large sample approximations to the above distributions are also presented.

Finally, considering general Σ_i's of equal dimensions, for H_{01} and H_{02}, it is shown that the "modified" LRT's are unbiased--a result parallel to Sugiura and Nagao (1968) on tests of covariance matrices.

2. Applications.

In addition to the mathematically interesting nature of the problem and the applications cited above, there lies a rich, fertile area for numerous applications of the SGV's. In fact, wherever variance is employed for univariate situations, SGV's seem to be applicable for the multivariate situations of 'overall' variability. Some examples are cited below.

(a) Multivariate Quality Control. It is well known [e.g., see Steyn (1978)] that testing H_0: the population mean vector $\underset{\sim}{\mu}_r$ of $\underset{\sim}{X}$, $\underset{\sim}{X}_r \sim N_p(\underset{\sim}{\mu}_r, \Sigma)$, remains constant during the sampling process against the alternative that $\underset{\sim}{\mu}_r$ varies during the process, is equivalent to testing H_0: GV = $|\Sigma|$ against H_1: GV = $|\Sigma^*|$, where $\Sigma^* = (I+2D/n)\Sigma$, $D = \sum_{r=1}^{\hat{}} \underset{\sim}{\mu}_r \underset{\sim}{\mu}_r' \Sigma^{-1}$, n being the sample size. One of the many applications of this result and, hence, test for SGV can be seen in multivariate quality control.

(b) Generalized Canonical Variable (GCV) Analysis. When the p-component original vector can be divided into $k > 2$ mutually exclusive groups, Anderson (1958), Problem 5, pp. 305-306, proposed GCV's to be obtained by minimizing their GV. Steel (1951) and Kettenring (1971) (as in his Ph.D. thesis) have attempted to construct such GCV's. Sen Gupta (1980a) constructed GCV's which are equicorrelated and formulated and studied some inference problems associated with such GCV's. However, no results on statistical inference associated with the GCV's obtained by Anderson-Kettenring-Steel's method are available. If the original vector can be meaningfully subgrouped into $k_1 < k_2 < \ldots < k_s$ groups, then one is faced with the problem of choosing between canonical variables with dimensions k_i, $i = 1,\ldots,s$. For the k_i-th subgrouping, the original vector is split into k_i mutually exclusive and exhaustive classes, the j-th class consisting of p_j elements, $\sum_{j=1}^{k_i} p_j = p$, $i = 1,\ldots,s$. Obviously, since the criterion of minimum GV is being used to represent the original vector by a smaller dimensional one, a GCV with the smallest dimension k_1, will be preferred provided they all have the same comparable GV's, i.e., the same SGV's. Also, it may be meaningful to represent the original vector by different GCV's of the same dimensions, k, but whose components are derived from different subgroupings of the original vector. Gnanadesikan (1977), pp. 74-77, drawing from a well-known example in psychometry, considers three (=k) sets of scores by several people on three batteries of three tests each, i.e., $p_1 = p_2 = p_3 = 3$. The three tests in each battery were intended to measure, respectively, the verbal, the numerical, and the spatial abilities of the persons tested. He commented, "An interesting alternative analysis in this example (...) would be to regroup the nine variables into three sets corresponding to the three abilities measured rather than the three batteries of tests." If the problem is considered in the more general set-up that the observed values merely constitute a sample from the underlying population, then how profitable the suggested alternative analysis will be must be judged through related statistical tests of significance. Furthermore, similar problems for a combination of the cases considered above, i.e., where one is faced with the problem of comparing several GCV's, some of which may possibly differ in dimensions, are quite important from practical considerations. Such statistical problems as comparing GCV's of possibly different dimensions obtained by Anderson-Kettenring-Steel's method can thus be formulated as tests for SGV's.

(c) Generalized Homogeneity of Multidimensional Scatter. Dyer and Keating (1980) were interested in the homogeneity of variances of sealed bids of five Texas offshore oil and gas leases. The variances were computed from the biddings of the oil companies, whose numbers and identities varied from lease to lease. In similar situations, if repeated observations are available for the leases or if the variabilities (for the bidding patterns) for certain companies for repeated bids over possibly different numbers of leases need to be compared, then one is faced with the problem of testing homogeneity of "variances," which we will term "Generalized Homogeneity" for vector variables of possibly different dimensions. This will be equivalent to testing homogeneity of SGV's, which is precisely H_{03} defined above.

(d) Missing Observations on a Random Vector. In cases of missing observations on part of a random vector [Dahiya and Korwar (1980), Goodman (1968)], generalized homogeneity as considered above can be quite useful for comparing multidimensional scatter. Suppose for an agricultural product P, there are q important characteristics with regard to variability.

Let the complete observation vector be $\underset{\sim}{X} = (\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_r)$, $\underset{\sim}{X}_i : q_i \times 1$, $i = 1,\ldots,r$, $\sum_1^r q_i = q$. Let observations be available from population j on only $t_j \le r$ of the $\underset{\sim}{X}_i$'s, say, with no loss of generality, on $X^*_{t_j} = (X_1,\ldots,X_{t_j})$, $j = 1,\ldots,k$. Since usually for an agricultural product P, the overall variability of the product can be represented in terms of the volume, one can use GV as a measure of this overall variability. In dealing with populations of maize and cotton, this notion was reflected by Goodman in his concluding remark: "The six cotton populations indicate even more clearly that the generalized variance is a useful measure of overall variability that merits futher investigation." Let it be desired to compare the multidimensional scatter of P over the k populations. Due to missing observations one possibility is to consider the random vector, say X^*_u, $u = \underset{j}{\mathrm{Min}}\ t_j$ for which observations are available from all the populations and compare the GV's of X^*_u. This was the approach of Goodman's analysis where "Only plants for which complete data were available were used for these analyses." This approach however results in loss of data. The other and better alternative, in case the missing observations are in the format described above would be to compare the SGVs of the $X^*_{t_j}$. [If measurements on the components are in different units, we can consider the standardized scatter coefficient, where scatter coefficient, defined by Frisch (1929), is the determinant of the correlation matrix]. This seems reasonable since more characteristics related to the variability of P is considered through $X^*_{t_j}$'s.

(e) Ranking and Selection. The method of ranking and selection based on GVs as proposed by Gnanadesikan and Gupta (1970) for equi-dimensional vector variables has a natural extension to an initial test for SGVs for vector variables differing in dimension followed by ranking and selection procedures based on the SGVs.

(f) Cluster Analysis. Often in cluster analysis, after the clusters are identified the question of homogeneity of clusters is of vital importance, as in Mezzich and Solomon (1980). Tests for SGVs can be exploited advantageously for such circumstances.

3. Likelihood Ratio Tests for SGV's.

Let $X \sim N_p(\mu,\Sigma)$. Throughout our discussion, unless otherwise stated we will assume Σ to be non-singular. Denote the population SGV of X, $|\Sigma|^{1/p}$ by Δ^2 and that of the sample, $|S/N|^{1/p}$ by d^2 where S is the sample sums of products matrix based on a sample of size N. Also denote $|S|^{1/p}$ by s^2. [Note that, Anderson (1958) defines GV with the divisor N-1 instead of N].

3.1. Test for a Specified Value of SGV. Let $x_1,\ldots,x_N$ be a random sample from $N_p(\mu,\Sigma)$ and suppose we want to test H_0: $\Delta^2 = \sigma_0^2$ (specified) against H_1: $\Delta^2 \neq \sigma_0^2$. (Note that H_0 is equivalent to the hypothesis that the GV, $|\Sigma|$, has the specified value σ_0^{2p}). Since the H_0 does not constrain μ, we have $\hat{\mu} = \bar{x}$. To find the MLE of Σ under H_0, consider

$$\Phi = \ln C + \sum_{i=1}^{p} \left(\frac{N}{2} \ln \theta_i - \frac{\theta_i}{2}\right) + \frac{\lambda}{2} \left(\ln s^{2p} - \sum_{i=1}^{p} \ln \theta_i - \ln \sigma_0^{2p}\right) ,$$

where $C = (2\Pi)^{-Np/2}|S|^{-N/2}$, λ is the Lagrange undetermined multiplier, θ_i, $i = 1, \ldots, p$ are the characteristic roots of $\Sigma^{-1}S$ and we have used the fact that $|\Sigma|^{1/p} = \sigma_0^2$ is equivalent to $\ln s^{2p} - \sum_{i=1}^{p} \ln \theta_i = \ln \sigma_0^{2p}$, $s^{2p} = |S|$. Differentiating Φ w.r.t. θ_i and equating to zero, we have $N - \lambda = \theta_i$, $i = 1,\ldots,p$. So,

$$(N-\lambda)^p = s^{2p}/\sigma_0^{2p} \Rightarrow \theta_i = s^2/\sigma_0^2\,, \qquad i = 1,\ldots,p\ .$$

Hence,

$$L_\omega/L_\Omega = C_1 a^{N/2} \exp(-p/2\ a^{1/p}) = f(a), \text{ say, where } C_1 = (e/N)^{np/2} \text{ and } a = s^{2p}/\sigma_0^{2p}\ .$$

However,

$$f(a) \uparrow a < N^p \quad \text{and} \quad \downarrow \ a > N^p\ .$$

So, we get,

Result 1. The LRT for H_0: $\Delta^2 = \sigma_0^2$ against H_1: $\Delta^2 \neq \sigma_0^2$ can be equivalently given by,

$$\text{Reject } H_0 \text{ iff } d^{2p}/\sigma_0^{2p} > a_0 \text{ or } < a_1\ ,$$

where a_0 and a_1 are constants to be determined from the specified level of the test.

3.2. Test for the Equality of the SGVs of Two Independent Multivariate Normal Populations. Let x_u, $u = 1,\ldots,N_1$ and y_v, $v = 1,\ldots,N_2$ denote two independent random samples from $N_{p_i}(\mu_i,\Sigma_i)$, $i = 1,2$ respectively. We are interested in testing H_0: $\Delta_1^2 = \Delta_2^2$ against H_1: $\Delta_1^2 \neq \Delta_2^2$. where Δ_i^2, $i = 1,2$ are the population SGVs of X and Y respectively. As in Section 3.1, $\hat{\mu}_1 = \bar{x}$, $\hat{\mu}_2 = \bar{y}$. Let θ_i, $i = 1,\ldots,p_1$ and η_j, $j = 1,\ldots,p_2$ be the characteristic roots of $\Sigma_1^{-1}S_1$ and $\Sigma_2^{-1}S_2$ respectively where S_i, $i = 1,2$ are the sample sums of products matrices for X and Y respectively. For finding the MLEs of Σ_is under H_0, consider as before,

$$\Phi = C + \sum_{i=1}^{p_1}\left(\frac{N_1}{2}\,\ell n\,\theta_i - \frac{1}{2}\theta_i\right) + \sum_{j=1}^{p_2}\left(\frac{N_2}{2}\,\ell n\,\eta_j - \frac{1}{2}\eta_j\right) + \frac{\lambda}{2}\left[\frac{1}{p_1}\left(\ell n\, s_1^{2p_1} - \sum_i \ell n\,\theta_i\right)\right.$$

$$\left. - \frac{1}{p_2}\left(\ell n\, s_2^{2p_2} - \sum_j \ell n\,\eta_j\right)\right]$$

where C is a constant, λ is the Lagrange undetermined multiplier, $s_i^{2p_i} = |S_i|$, $i = 1,2$ and we have used the fact that "$|\Sigma_1|^{1/p_1} = |\Sigma_2|^{1/p_2}$" is equivalent to the quantity within the third bracket on the right-hand side of (3.1) being zero.

Differentiating w.r.t. θ_is and η_js and equating these to zeros, we have

$$p_1N_1 - \lambda = p_1\theta_i\ ,\ i = 1,\ldots,p_1 \Longrightarrow p_1N_1 - \lambda = p_1\left(s_1^{2p_1}/\sigma_0^{2p_1}\right)^{1/p_1}$$

and

$$p_2N_2 + \lambda = p_2\eta_j\ ,\ j = 1,\ldots,p_2 \Longrightarrow p_2N_2 + \lambda = p_2\left(s_2^{2p_2}/\sigma_0^{2p_2}\right)^{1/p_2}\ .$$

These give $\hat{\sigma}_0^2 = (p_1s_1^2 + p_2s_2^2)/(p_1N_1 + p_2N_2) = (p_1N_1d_1^2 + p_2N_2d_2^2)/(p_1N_1 + p_2N_2)$ where d_i^2, $i = 1,2$ are the sample generalized variances for X and Y respectively and $\Delta_1^2 = \Delta_2^2 = \sigma_0^2$, the common unknown value. Note that σ_0^2, the MLE of σ_0^2 agrees with that in the case of $p_1 = p_2 = 1$. Also $\sigma_0^{2p_1}$ and $\sigma_0^{2p_2}$ give the MLEs of $|\Sigma_1|$ and $|\Sigma_2|$, the generalized variances of X and Y respectively. Using these estimates, the Likelihood Ratio criterion is given by,

$$(3.2.1) \qquad L_\omega / L_\Omega = \left\{ (s_1^2)^{N_1p_1/2} (s_2^2)^{N_2p_2/2} \right\} / (\hat{\sigma}_0^2)^{(N_1p_1+N_2p_2)/2}$$

$$= C.\ a^{N_2p_2/2} (p_1a+p_2)^{(N_1p_1+N_2p_2)/2}$$

$$= f(a), \text{ say, where } a = s_1^2/s_2^2 \text{ and } C \text{ is a constant.}$$

However, $f(a) \downarrow a > N_2p_2^2$ and $\uparrow 0 < a < 0\ N_2p_2^2/N_1p_1^2$. Thus

Result 2. The LRT for H_0: $\Delta_1^2 = \Delta_2^2$ against H_i: $\Delta_1^2 \neq \Delta_2^2$ can be equivalently given by,

$$\text{Reject } H_0 \text{ iff } R = d_1^2/d_2^2 < r_1 \text{ or } > r_2$$

where r_1 and r_2 are constants to be determined from the specified level of the test.

3.3. Test for the Equality of SGVs of $k(> 2)$ Independent Multivariate Normal Populations. Let $x_{i\ell}$, $\ell = 1,\ldots,N_i$, $i = 1,\ldots,k$ denote k random samples from k independent populations $N_{p_i}(\mu_i,\Sigma_i)$, $i = 1,\ldots,k$ respectively. We are interested in testing H_0: Δ_i^2 , $i = 1,\ldots,k$ all equal, against the alternative H_1, that at least one of them differ. As before, under both H_0 and H_1 , $\hat{\mu}_i = \bar{x}_i$, $i = 1,\ldots,k$. Let θ_{ij} , $i = 1,\ldots,k$, $j = 1,\ldots,p_i$ be the characteristic roots of $\Sigma_i^{-1}S_i$ respectively where S_i, $i = 1,\ldots,k$ are the sample sums of products matrices for X_i , $i = 1,\ldots,k$ respectively. For finding the MLEs of Σ_i , $i = 1,\ldots,k$, under H_0 , consider as before,

(3.3.1)

$$\Phi = C + \sum_{i=1}^{k} \sum_{j=1}^{p_i} \left\{ \frac{N_i}{2} \ln \theta_{ij} - \frac{1}{2} \theta_{ij} \right\}$$

$$+ \sum_{i=1}^{k} \frac{1}{2} \lambda_{ii+1} \left(\left[\sum_j \frac{1}{p_i} \ln \theta_{ij} - \ln s_i^2 \right] - \left[\sum_j \frac{1}{p_{i+1}} \ln \theta_{i+1j} - \ln s_{i+1}^2 \right] \right]$$

where C is a constant and λ_{ii+1} are undetermined Lagrange multipliers with $k+1$ being replaced by 1 in the suffixes. Differentiating ϕ with respect to θ_{ij}s and equating to zeros we have,

$$p_i N_i + (\lambda_{ii+1} - \lambda_{i-1i}) = p_i \theta_{ij} \; ; \; i = 1, \ldots, k, \;\; j = 1, \ldots, p_i \;, \; \lambda_{01} = \lambda_{k1}$$

$$\Rightarrow \theta_{ij} = \theta_{ij'} \;, \;\; \Rightarrow \;\; \theta_{ij} = s_i^2/\hat{\sigma}_0^2 \;, \; i = 1, \ldots, k \;,$$

where $\hat{\sigma}_0^2$ is the MLE of σ_0^2, the common unknown value of Δ_i^2, $i = 1, \ldots, k$. So

$$\hat{\sigma}_0^2 \, \Sigma \, p_i N_i + \hat{\sigma}_0^2 \, [\sum_{i=1}^{k} (\lambda_{ii+1} - \lambda_{i-1i})] = \Sigma \, p_i s_i^2$$

$$\Rightarrow \hat{\sigma}_0^2 = \sum_{.i.} p_i s_i^2 / \sum_i p_i N_i \; .$$

Note again this agrees with the MLE for σ_0^2 of the univariate case. Hence, we get,

<u>Result 3</u>. The LRT for H_0: Δ_i^2, all equal, against H_1: at least one of the Δ_i^2, $i = 1, \ldots, k$, differ is given by

$$\text{Reject } H_0 \text{ if and only if }, \; \eta = \prod_{i=1}^{k} (d_i^2/\hat{\sigma}_0^2)^{N_i p_i/2} < \eta_0$$

where η_0 is a constant to be determined from the specified level of the test.

4. Exact Null and Non-Null Distributions of the Test Criteria.

4.1.1. Definitions and Decisions. In order to obtain the exact distributions under null and alternative hypotheses, for the test statistics considered in Section 3, we recall the following definitions and preliminary discussions from Mathai [(1970), (1972), (1973)].

Meijer's G-Function. The G-function is defined as,

$$G_{p,q}^{m,n}\left\{z\,\middle|\,\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right\} = (2\pi i)^{-1}\int_L h(s)\, z^{-s}\, ds \,, \tag{4.1.1}$$

where $i = (-1)^{1/2}$, z is not equal to zero and

$$z^2 = \exp\{s\,(\ln|z| + i \arg z)\}$$

in which $\ln |z|$ denotes the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value, and

$$h(s) = \prod_{j=1}^{n}\Gamma(1-a_j-s)\prod_{j=1}^{m}\Gamma(b_j+s)/\{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)\}$$

where m, n, p, q are integers such that, $0 \le n \le p$, $1 \le m \le q$

$$a_j (j = 1,\dots,p) \;,\; b_j (j = 1,\dots,q)$$

are complex numbers such that

$$(b_h + v) \neq (a_j - 1 - r) \ , \quad \text{for} \quad v,\ r = 0,1,\ldots,$$

L is a contour in the complex s-plane such that the points,

$$-s = (b_j + v) \ , \ j = 1,\ldots,m \ ; \ v = 0,1,\ldots \tag{4.1.2}$$

and

$$-s = (a_j - 1 - v) \ , \ j = 1,\ldots,n \ ; \ v = 0,1,\ldots$$

are separated and the points in (4.1.2) are enclosed by L . An empty product is interpreted as unity. The function (4.1.1) makes sense in the following cases

For every $z \neq 0$ if $q - p$ is positive and for $0 < |z| < 1$ if $q - p = 0$. The existence of different contours L is discussed in Erdélyi (1953, p. 207).

The Gauss-Legendre Multiplication Formula.

$$\Gamma(m z) = (2\pi)^{(1-m)/2} \, m^{mz-1/2} \prod_{j=0}^{m-1} \Gamma(z + j/m) \ , \ m = 1,2,\ldots \ . \tag{4.1.3}$$

This formula (4.1.3) enables one to write a Gamma of an integral multiple of z in terms of Gamma of z.

The technique of inverse Mellin transform can be advantageously employed when a moment sequence uniquely determines a distribution.

Rao (1973, p.106) has given a number of sufficient conditions for the unique existence of the density function. If μ_{s-1}, $s = 0,1,\ldots$ is the (s-1)th moment about the origin of T, uniquely determining its density, then the density $g(t)$ of T is given by the inverse Mellin transform

$$g(t) = (2\pi i)^{-1} \int_{C+i\infty}^{C+i\infty} \mu_{s-1}\, t^{-s}\, ds \tag{4.1.4}$$

where $i = \sqrt{-1}$ and C is a suitably chosen real number. For multivariate normal populations it is seen that at least one of the conditions cited in Rao is satisfied by the moment sequence.

A computable representation of the G-function of (4.1.1) using the Calculus of Residues is given by,

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = \sum_{k=1}^{n} \sum_{j=1}^{q_k} A_{kj} + \sum_{r=1}^{t} \sum_{j=1}^{n_r} \sum_{v=1}^{r} \sum_{i=0}^{m_{jrr}-v-1} R_{vi}\,, \tag{4.1.5}$$

where A_{kj} is given in (5.21) and R_{vi} is given in (5.27) of Mathai (1970, p. 141).

<u>Pincherle's H-Function</u>. The G-function is a special case of the most general Special Function, namely the H-function. Braaksma (1964) has discussed the H-function in detail. Following Mathai, slight modifications of the original definition are made here in order to represent it as an inverse Mellin transform. We define the H-function as follows.

$$(4.1.6) \qquad H(z) = H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{n=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}\, z^{-s}\, ds$$

where $i = \sqrt{-1}$, p, q, m, n are integers such that $1 \leq m \leq q$, $0 \leq n \leq p$, α_j $(j = 1,\ldots,p)$, β_j $(j = 1,\ldots,q)$ are positive numbers and a_j $(j = 1,\ldots,p)$, b_j $(j = 1,\ldots,q)$ are complex numbers such that,

$$\alpha_j(b_n+\nu) \neq \beta_h(a_j-1-\lambda) \text{ , for } \nu,\lambda = 0,1,\ldots,;\ h = 1,\ldots,m\ ;\ j = 1,\ldots,n\ .$$

L is a contour separating the points,

$$(4.1.7) \qquad -s = (b_j+\nu)/\beta_j \text{ , } j = 1,\ldots,m;\ \nu = 0,1,\ldots$$

and

$$(4.1.8) \qquad -s = (a_j-1-\nu)/\alpha_j \text{ , } j = 1,\ldots,n\ ;\ \nu = 0,1,\ldots\ .$$

One condition of existence of the H-function is that it exists, for every $z \neq 0$ if $\mu > 0$, where

$$\mu = \sum_1^q \beta_j - \sum_1^p \alpha_j$$

and for $0 < |z| < \beta^{-1}$, if $\mu = 0$ where

$$\beta = \prod_{1}^{p} \alpha_j^{\alpha_j} \prod_{1}^{q} \beta_j^{-\beta_j} .$$

The result due to Braaksma [p. 278, (6.1)] which effectively says that $H(z)$ is available as the sum of the residues of $h(s) z^{-s}$ in the points (4.1.7) is not affected by the modification mentioned earlier of the original definition of the H-function. A computable representation of the H-function, with the detailed method of identifying the poles is given in Mathai (1973) as follows.

$$H(z) = \sum_{j=1}^{m} \sum R_j \tag{4.1.9}$$

where the second sum is over $S_{j_1 \cdots j_m}^{(jj)}$ defined in (4.10) and R_j is defined in (4.22) of Mathai (1973).

<u>4.2. Exact Distribution of d^{2p}/σ_0^{2p}</u>. Since the sample generalized variance d^{2p}, arise frequently in many multivariate tests, various authors, e.g. Mathai (1970), Consul (1964), have worked on its distribution. Mathai's work seems to be quite suitable in view of the distribution that will be needed for Section 4.3. Letting $v = d^{2p}/\sigma_0^{2p}$ and using (4.1.4), the density of v, $g(v)$, under H_0, can be written as

$$g(v) = C(2\pi i)^{-1} \int_L \prod_{j=1}^{p} \Gamma[(N-j)/2+h] v^{-s} ds$$

$$= C\, G_{0,p}^{p,0}(v \mid (N-j)/2 \ , \ j = 1,2,\ldots,p) \ , \ 0 < v < \infty \tag{4.2.1}$$

where $\quad C^{-1} = \prod_{j=1}^{p} \Gamma[(N-j)/2]$

A computable representation of $g_1(v)$ is then obtained by substituting $m = p$, $n = 0$, $p = 0$, $q = p$ in (4.1.5)

Explicit evaluations for the residues needed in (4.1.5) are given in Mathai (1972). For this, Mathai considered a random variable with its h-th moment given by,

$$(4.2.2) \qquad E(X^h) = \left(\frac{\beta}{\alpha}\right)^h \prod_{i=1}^{p} \frac{\Gamma[(m-i)/2+h]}{\Gamma[(n-i)/2]} \prod_{i=1}^{p'} \frac{\Gamma[(n-i)/2-h]}{\Gamma[(m-i)/2]}$$

Assume the moment sequence in (4.2.2) uniquely determines the density of X. Then, through the use of Calculus of Residues he obtained the following

<u>Theorem M1</u>. The density function $f(x)$ corresponding to the moment sequence in (4.2.2) is as follows.

$$f(x) = C(\alpha x/\beta)^{m/2-(p+1)/2} x^{-1} \sum_{j=1}^{\infty} (R_j + R_j') \ , \ x > 0 \ .$$

$$R_j = [a_j - 1)!]^{-1} (\alpha x/\beta)^j \sum_{r=0}^{a_j-1} \binom{a_j-1}{r} (-\log \alpha x/\beta)^{a_j-1-r}$$

$$\times \left\{ \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} \times A^{(r-1-r_1)} \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} A^{(r_1-1-r_2)} \right\} B \ ,$$

R_j' is R_j with $(\alpha x/\beta)^j$, a_j, $A^{(t)}$, B are replaced by $(\alpha x/\beta)^{j-1/2}$, b_j, $A'(t)$ and B' respectively. The various quantities for the two cases p-even and p-odd are given in pp. 166-167 of Mathai (1972).

For v, the moment sequence given in (4.2.2) determines the density uniquely according to Rao (1973, p. 106). Letting $p' = 0$,

$\alpha = 1$, $\beta = 2^p|\Sigma|/\sigma_0^{2p}$ and $m = n$, the density of v is deduced in a form suitable for computation. Using the specified values for $|\Sigma|$ under the null $(\beta = 2^p)$ and alternative hypotheses, the corresponding distributions of v are obtained.

4.3. Exact Distribution of R. Because of its frequent applications in various multivariate test procedures, distribution of the ratio of two independent sample GVs have received much attention, e.g. Nandi (1977), Tretter and Walster (1975), Mathai (1972) etc.

Case 1. $p_1 = p_2$. The test statistic R will be equivalent to the test statistic based on the ratio of two independent GVs, say, w, only when $p_1 = p_2 = p$, say. In that case, the density of $w = R^p$ is obtained from Theorem M1 above by letting $\alpha = 1/\Delta_1^2$ and $\beta = 1/\Delta_2^2$. Under H_0 $\alpha/\beta = 1$. Under any alternative H: $\Delta_1^2/\Delta_2^2 = \delta^2$, the distribution of R is obtained by substituting $\alpha/\beta = \Delta_2^2/\Delta_1^2 = 1/\delta^2$ in Theorem M1.

Case 2. $p_1 \neq p_2$. In the case of unequal dimensions, the distribution of R is not available. We obtain the distribution in terms of the H-function and present it in a computable form through the use of Calculus of Residues. Now,

$$(4.3.1) \qquad E(R^h) = C\,(\Delta_1^2/\Delta_2^2)^h \prod_1^{p_1} \Gamma\{\tfrac{1}{2}(N_1 + \tfrac{2h}{p_1} - i)\} \prod_1^{p_2} \Gamma\{\tfrac{1}{2}(N_2 - \tfrac{2h}{p_2} - i)\}$$

where

$$C = [\prod_1^k \Gamma\{(N_1-i)/2\} \prod_1^{p_2} \Gamma\{(N_2-i)/2\}] .$$

Hence from (4.1.4), on using (4.1.6), the density of R, $g_2(r)$, under H_0, can be written as

$$(4.3.2) \qquad g_2(r) = (2\pi i)^{-1} \int_L E(R^h)\, r^{-h}\, dh$$

$$= C_1 \cdot H^{p_1,p_2}_{p_2,p_1}\left[r\left|\begin{matrix}(a_1,p_1),\ldots,(a_{p_2},p_1)\\(b_1,p_2),\ldots,(b_{p_1},p_2)\end{matrix}\right.\right], \quad 0 < r < \infty .$$

Since here the α_js and the β_js in (4.1.6) corresponding to (4.3.1) are all rational numbers, we can use (4.1.3) to express $g_2(r)$ in terms of G-function also. However, use of H-function here is a more direct and convenient approach.

In order to present (4.3.2) in a computable form we introduce the following notation. Consider (4.1.6). The poles of $\Gamma(b_j+\beta_j s)$ are given by the equation

$$(4.3.3) \qquad -s = (b_j+\nu)/\beta_j \ , \quad \nu = 0,1,\ldots .$$

If the point $-s = (b_j+\nu_1)\beta_j$ for some $\nu = \nu_1 \varepsilon \{0,1,\ldots\}$ coincides with the poles coming from $\gamma-1$ other Gammas of the set $\Gamma(b_j+\beta_j s)$, $j = 1,\ldots,m$ then the point gives a pole of order γ. In order to distinguish poles of all orders, for a fixed j, consider the equations

$$(4.3.4) \qquad \frac{b_1+\nu^{(j1)}_{j_1\cdots j_m}}{\beta_1} = \frac{b_2+\nu^{(j2)}_{j_1\cdots j_m}}{\beta_2} = \cdots = \frac{b_m+\nu^{(jm)}_{j_1\cdots j_m}}{\beta_m} \cdots .$$

We interpret (4.3.4) as follows. For a fixed j, $j_r = 0$ or 1 for $r = 1,\ldots,m$. If $j_r = 0$, then $(b_r + \nu^{(jr)}_{j_1\cdots j_m})/\beta_r$ is to be excluded from (4.3.4) $\nu^{(jj)}_{j_1\cdots j_m}$ is a value of ν in (4.3.3). For every fixed j, $j_1 + \cdots + j_m \in \{1,\ldots,m\}$ denotes the order of the pole at $-s = (b_j + \nu^{(jj)}_{j_1\cdots j_m})/\beta_j$. To avoid duplication we will use the convention, while considering the points corresponding to (4.3.3), that

$$j_1 = j_2 = \cdots = j_{j-1} = 0.$$

Denote by $S^{(jj)}_{j_1\cdots j_m}$ the set of all values $\nu^{(jj)}_{j_1\cdots j_m}$ takes for given $j_1, j_2, \ldots, j_m$, i.e. $S^{(jj)}_{j_1} = \{\nu^{(jj)}_{j_1\ldots j_m}\}$. Then

$$H(z) = \sum_{j=1}^{m} \sum_{S^{(jj)}_{j_1\cdots j_m}} R_j \tag{4.3.5}$$

where R_j is the residue of $h(s)z^{-s}$ at the pole $-s = (b_j + \nu^{(jj)}_{j_1\cdots j_m})/\beta_j$ and $\sum_{S^{(jj)}_{j_1\ldots j_m}}$ denotes the summation over all sets $S^{(jj)}_{j_1\cdots j_m}$ denotes the summation over all sets $S^{(jj)}_{j_1\cdots j_m}$. More detailed discussions can be found in Mathai (1973).

We now determine the poles and their corresponding orders needed in (4.3.5), i.e. only the first product of gammas in (4.3.1). It will thus be convenient, for computational purposes, to choose $p_1 < p_2$.

For a fixed i, the poles of $\Gamma[\frac{N_1 - i}{2} + \frac{h}{p_1}] = \Gamma_i$ are given by the equation

$$-s = p_1\{(N_1 - i)/2 + \nu\}, \quad \nu = 0,1,2,\ldots .$$

Note that poles of Γ_i and Γ_j coincide only when i and j are both even or both odd.

Lemma 4.3.1. The poles, with their corresponding orders, are given by

Case A. p_1 odd.

(4.3.6) $\{\nu^{(11)}_{1010\cdots101}\} = \{0,1,2,\ldots\} = \{\nu\}$. Poles are $p_1(\frac{N_1-1}{2}+\nu)$,

repeated $\frac{p_1+1}{2}$ times .

$\{\nu^{(jj)}_{\underbrace{00..}_{j-1}0101\cdots101}\} = \{0\}$. Pole is $p_1(\frac{N_{1-j}}{2})$,

repeated $\frac{p_1+1}{2} - \frac{j-1}{2}$ times, $j = 3,5,\ldots,p_1$.

$\{\nu^{(22)}_{0101\cdots010}\} = \{0,1,2,\ldots\} = \{\nu\}$. Poles are $p_1(\frac{N_1-2}{2}+\nu)$,

repeated $\frac{p_1-1}{2}$ times .

$\{\nu^{(jj)}_{\underbrace{0\cdots}_{j-1}010..010}\} = \{0\}$. Pole is $p_1(\frac{N_1-j}{2})$,

repeated $\frac{p_1-1}{2} - \frac{j-2}{2}$ times , $j = 4,6,\ldots,p_1-1$.

Case B. p_1 even.

(4.3.7) $\{\nu^{(11)}_{1010\cdots10}\} = \{0,1,2,\ldots\} = \{\nu\}$. Poles are $p_1(\frac{N_1-1}{2}+\nu)$,

repeated $\frac{p_1}{2}$ times .

$\{\nu^{(jj)}_{0\cdot\cdot010\cdot\cdot10}\} = \{\nu\}$. Pole is $p_1(\frac{N_1-j}{2})$,

repeated $\frac{p_1}{2} - \frac{j-1}{2}$ times, $j = 3,5,\ldots,p_1-1$.

$\{\nu^{(22)}_{0101\cdots01}\} = \{0,1,2,\ldots\} = \{\nu\}$. Poles are $p_1(\frac{N_1-2}{2}+\nu)$,

repeated $\frac{p_1}{2}$ times.

$\{\nu^{(jj)}_{00\cdots1..01}\} = \{0\}$. Pole is $p_1(\frac{N_1-j}{2})$,

repeated $\frac{p_1}{2} - \frac{j-2}{2}$ times, $j = 4,6,\ldots,p_1$.

For $i \neq j$, $\{\nu^{(ij)}\}$ is vacuous unless i and j are both odd or both even. (We omit the subscripts of $\nu^{(ij)}$ since it is clear what they are.)

(4.3.8) $\{\nu^{(1\ell)}\} = \{\nu + \frac{\ell-1}{2}\}$, $\nu = 0,1,2,\ldots;\ell > 1$ is odd. Poles are identified with those of $\nu^{(11)}$.

$\{\nu^{(2\ell)}\} = \{\nu + \frac{\ell-2}{2}\}$, $\nu = 0,1,2,\ldots;\ell > 2$ is even. Poles are identified with those of $\nu^{(22)}$.

$\{\nu^{(\ell'\ell)}\} = \frac{\ell-\ell'}{2}$, $\ell > \ell' > 2$; $\ell'\ell$ both odd or both even. Poles are identified with those of $\nu^{(\ell\ell)}$.

Proof. The above results follow from the following observations. Consider Case A. Let $i = 1$ and $j \leq p_1$ be any odd number. Then, poles of Γ_1 and Γ_j coincide, as

$$p_1(\frac{N_1-1}{2} + \nu) = p_1(\frac{N_1-j}{2} + \lambda) \Rightarrow (\nu,\lambda) = \{(0, \frac{j-1}{2}),(1, \frac{j+1}{2}),\ldots\}$$

But this set excludes the poles coming from $\lambda \in \{0,1,\ldots,\frac{j-1}{2}-1\} = E_j$. COnsider j,j' both odd, $3 \le j < j' \le p_1$. Then

$$(4.3.9) \qquad p_1\left(\frac{N_1-j}{2}+\lambda\right) = p_1\left(\frac{N_1-j'}{2}+\lambda'\right). \text{ For } \lambda = \frac{j-1}{2}-1+\frac{j'-j}{2} .$$

Thus considering the 'excluded sets' E_js we note that the smallest element, i.e. 0 , is repeated in all succeeding E_j, through the relation (4.3.9). This establishes (4.3.6). A similar argument holds for (4.3.7). (4.3.8) follows from the definitions of the corresponding sets.

Theorem 4.3.1. The probability density function of R is given by for p_1 odd,

$$g_2(r) = C\cdot\Bigg[\sum_{\nu=0}^{\infty} (r/\delta^2)^{p_1\left(\frac{n_1-1}{2}+\nu\right)} \sum_{u=0}^{(p_1-1)/2^{-1}} f(r/\delta^2;u,a_1,A_0,B_0)$$

$$+\sum_j^{*} (r/\delta^2)^{p_1(N_1-j)/2} \sum_{u=0}^{\frac{p_1+1}{2}-\frac{j-1}{2}-1} f(r/\delta^2;u,a_j,A_0,B_0)$$

$$+\sum_{\nu=0}^{\infty} (r/\delta^2)^{p_1\left(\frac{N_1-2}{2}+\nu\right)} \sum_{u=0}^{(p_1-1)/2^{-1}} f(r/\delta^2;u,b_1,A_0,B_0)$$

$$+\sum_j^{**} (r/\delta^2)^{p_1(N_1-j)/2} \sum_{u=0}^{\frac{p-1}{2}-\frac{j-2}{2}-1} f(r/\delta^2;u,b_j,A_0,B_0)\Bigg],$$

and for p_1 even

$$g_2(r) = C\left[\sum_{\nu=0}^{\infty} (r/\delta^2)^{p_1(\frac{N_1-1}{2}+\nu)} \sum_{u=0}^{\frac{p_1}{2}-1} f(r/\delta^2\,;\,u,a_1',A_0,B_0)\right.$$

$$+ \sum_j{}^{*} (r/\delta^2)^{p_1(N_1-j)/2} \sum_{u=0}^{\frac{p_1}{2}-\frac{j-1}{2}-1} f(r/\delta^2\,;\,u,a_j',A_0,B_0)$$

$$+ \sum_{\nu=0}^{\infty} (r/\delta^2)^{p_1(\frac{N_1-2}{2}+\nu)} \sum_{u=0}^{\frac{p_1}{2}-1} f(r/\delta^2\,;\,u,b_1,A_0,B_0)$$

$$\left.+ \sum_j{}^{**} (r/\delta^2)^{p_1(N_1-j)/2} \sum_{u=0}^{\frac{p_1}{2}-\frac{j-2}{2}-1} f(r/\delta^2\,;\,u,b_j,A_0,B_0)\right]$$

where

$$f(r:u,d,A_0,B_0) = \frac{1}{(u_0-1)!} \begin{pmatrix} u_0-1 \\ u \end{pmatrix} (-\log r)^{u_0-1-u} \left[\sum_{\gamma_1=0}^{u-1} \binom{u-1}{1} A_0^{(u-1-\gamma_1)}\right.$$

$$\left.\times \sum_{\gamma_2=0}^{\gamma_1-1} \begin{pmatrix} \gamma_1-1 \\ \gamma_2 \end{pmatrix} A_0^{(\gamma_1-1-\gamma_2)} \cdots B_0\right]$$

$$B_0 = (s-d)^{u_0} \prod_{1}^{p_1} \Gamma\{\tfrac{1}{2}(N_1 + \frac{2h}{p_1} - i)\} \quad \text{at} \quad s = d$$

where d is a pole of order u_0, (the upper limit +1, for u in the summation in the theorem) of the product of the gamma functions defined by B_0.

$$A_0^{(t)} = \frac{\delta^{t+1}}{\delta s^{t+1}} \log B_0\,, \quad t \geq 0\,.$$

C is the constant defined in (4.3.1) and $\delta^2 = \Delta_1^2/\Delta_2^2$. Σ_j^* and Σ_j^{**} denote the summations over all $j \varepsilon \{3,5,\ldots,p_1^*\}$ and $j \varepsilon \{4,6,\ldots,p_1^{**}\}$ respectively, $p_1^* = p_1$ if p_1 is odd and $p_1^* = p_1-1$ if p_1 is even; $p_1^{**} = p_1$ if p_1 is even and p_1-1 if p_1 is odd.

Proof. The proof follows by noting (4.3.5) and combining Lemma 4.3.1 above with Lemma 1 of Mathai (1973).

Finally, a convenient computational form of the p.d.f. of R is obtained from the following theorem proved as Theorem 1 in Mathai (1973).

Theorem M2. $H(z)$ is given in (4.3.5) where,

$$(4.3.10) \qquad R_j = \frac{(j_1+\cdots+j_m)\, z^{(b_j+v^{(jj)}_{j_1\cdots j_m})/\beta_j}}{(j_1+\cdots+j_m)!} \sum_{r=0}^{j_1+\cdots+j_m-1} \binom{j_1+\cdots+j_m-1}{r}$$

$$\times (-\log z)^{j_1+\cdots j_m-1-r} \left[\sum_{r_1=0}^{r-1} \binom{r-1}{r_1} C_j^{(r-1-r_1)} \right.$$

$$\left. \times \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} C_j^{(r_1-1-r_2)} \cdots \right] D_j \ ,$$

where the C_js and D_js are defined in (4.23) and (4.24) of Mathai (1973).

We note that the sets $\{\nu^{(jh)}_{j_1\cdots j_m}\}$ are not needed for $h < j$ in (4.23) and (4.24) of Mathai. Thus, Lemma 4.3.1 gives us all the desired sets needed to use Theorem M2 above, which expresses $H(z)$ in terms of the convenient computable functions, e.g. the Psi and the generalized zeta functions. Examples of computation of $H(z)$ is given in Section 5 of Mathai (1973). Also computational procedure and computer programs for calculating the percentage points of the distribution of R can be obtained in a manner similar to Mathai and Katiyar (1979). The null distribution is obtained by putting $\delta^2 = \Delta_1^2/\Delta_2^2 = 1$ in (4.3.1) and the nonnull distribution by substituting the specified value, under the given alternative, $\delta^2 = \Delta_1^2/\Delta_2^2$ in (4.3.1). It is known that for $p_i = 1$ or 2, $X_j = p_j n_j u_j^2/\Delta_j^2$ where $n_j = N_j - 1$ and $n_j u_j^2 = N_j d_j^2$, $j = 1,2$ is distributed as a χ^2 with d.f. $p_j(n_j - p_j + 1)$. Hence if $p_i = 1$ or 2, p_1 not necessarily equal to p_2, the exact distribution of R, under both the null and alternative hypotheses are obtained as central F_{ξ_1,ξ_2} distributions, with obvious multipliers, having d.f. given by $\xi_i = p_i(n_i - p_i + 1)$, $i = 1,2$.

4.4. Exact Distribution of η. We consider a Bartlett type modification for η. Let $X_i = p_i n_i u_i^2/\Delta_0^2$ where $n_i = N_i - 1$ and $n_i u_i^2 = N_i d_i^2$, $i = 1,\ldots,k$. As in the univariate case, we propose the modified test statistic

$$\eta_B^2 = \prod_{i=1}^{k}\left(u_i^2\right)^{n_i p_i/\Sigma \nu_i p_i} \Bigg/ \left(\sum_{i=1}^{k} n_i\, p_i\, u_i^2 \Bigg/ \sum_{i=1}^{k} n_i\, p_i\right) \tag{4.4.1}$$

$$= C_b^{-1}\, \Pi X_i^{b_i} \Big/ \sum X_i$$

where

$$b_i = (n_i p_i / \textstyle\sum n_i p_i), \quad C_b = \Pi b_i^{b_i}.$$

For $p_i = 1$ or 2, $X_i \sim \chi^2_{p_i(n_i - p_i + 1)}$. Using this result and the representation (4.4.1) we get

Theorem 4.4.1. For $p_i = 1$ or 2, p_i not necessarily equal to p_j, $i \neq j, i,j = 1,\ldots,k$ the exact density of η_B^2 is given by

$$f(t) = \left[\Gamma(m) \Big/ \prod_1^k \Gamma(ma_j)\right] \left(\prod_1^k b_j^{a_j}\right)^m \left(\prod_1^k b_j^{-\frac{1}{2}}\right)$$

$$\cdot \left[(2\pi)^{(k-1)/2} \Big/ \Gamma((k-1)/2)\right] t^{m-1}(-\log t)^{(k-3)/2} \, \eta_{m,a,b}(t) ,$$

$$0 < t < 1$$

where,

$$m = \sum_{i=1}^k p_i(n_i - p_i + 1)/2, \quad a_j = p_j(n_j - p_j + 1)/2m, \; j=1,\ldots,k.$$

b_j, $j=1,\ldots,k$ are defined in (4.4.)

and $\eta_{m,a,b}$ is defined in Theorem 2 of Ghao and Glaser (1978).

Percentage points and approximations to the above distribution are obtained from Dyer and Keating (1980). For $p_i \geq 3$, the distributions of η^2 or η_B^2 seem to be complicated.

5. Large Sample Approximations. Some large sample approximations to the exact distributions of the test criteria considered above are now suggested. Existing approximations are also reviewed for the distribution of GV and SGV.

5.1 Asymptotic Distributions of GV and SGV. Letting $nu^2 = Nd^2$, $n = N-1$, we have from Anderson (1958), Theorem 7.5.4, that for large N,

$$\sqrt{n}\,(u^{2p}/\Delta^{2p} - 1) \xrightarrow{L} N(0,2p).$$

It is know that $\eta = N^p v = |S|/|\Sigma|$ is distributed as $\prod_{i=1}^p \chi^2_{N-i}$, where the χ^2's are all independent. Hoel (1937) suggested approximating the distribution of $\eta^{1/p} = w$ by the distribution with the density function

$$g(w) = \frac{C^{\frac{1}{2}p(N-p)} \; w^{\frac{1}{2}p(N-p) - 1} \; e^{-Cw}}{\Gamma[1/2p(N-p)]}$$

where,

$$C \equiv C(p,N) = \frac{p}{2}\left[\,1 - \frac{(p-1)(p-2)}{2N}\right]^{1/p}$$

This turns out to be exact for p=1 and p=2.

Gnanadesikan and Gupta (1970) have suggested approximating the distribution of $\ln w = \frac{1}{p}\ln \eta = \frac{1}{p}\sum^{p} \ln \chi^2_{N-i}$, using the central limit Theorem, by the normal distribution.

We now propose a new approximation to the distribution of SGV. An application of the general result of Madansky and Olkin (1969) with $h(V) = |V|^{1/p}$ shows that, for large N,

$$\sqrt{N}(d^2/\Delta^2 - 1) \xrightarrow{L} N(0,2/p).$$

In the light of this approximation to the distribution of the SGV, it is interesting to note the approximation to the distribution of GV by Anderson stated at the beginning. Little is known about the relative performances of the four approximations discussed above.

5.2 Asymptotic Distributions of R and η. Letting $c_i = C(N_i, p_i)$ of Hoel's approximation, for large N_1 and N_2, the density of R can be approximated by that of,

$$\frac{C_2 N_2 p_1 (N_1 - p_1)}{C_1 N_1 p_2 (N_2 - p_2)} \; \delta^2 F_{p_1(N_1-p_1),\, p_2(N_2-p_2)},$$

where $\delta^2 = \Delta_1^2/\Delta_2^2$. The null and non-null distributions are obtained by putting δ^2=1 and the specified value under the alternative hypothesis, respectively.

In addition to the usual χ^2 approximation to the likelihood ratio criterion η^2, another approximation is presented here. If N_i is large compared to p_i^2, $i=1,\ldots,k$ then in the same lines of Hoel's approximation, we get $X_i = p_i n_i u_i^2/\Delta_o^2$ can be approximated by a χ^2 variable with d.f. $p_i(N_i-p_i)$, $i=1,\ldots,k$. Hence,

Lemma 5.2.1. If N_i is large compared to $p_i^2, i=1,\ldots,k$, then the density of η_B^2 under Ho can be approximated by f(t) defined in Theorem 4.4.1 (where p_is now can be any integers, not necessarily 1s or 2s only).

Similar result is seen to hold for η^2 also. These approximations are expected to be better than the usual χ^2 approximations.

6. A Multivariate F_{max} Criterion. A simpler statistic than η^2 is now suggested for the special case when we have an equal number of observations, N, from k populations, each of equal dimension p, e.g.,

$$F_{p,max} = d^2_{max}/\, d^2_{min}$$

For p=1, this coincides with the F_{max} proposed by Hartley (1950) as a short cut method for the univariate case. It is known that $\ln \chi^2_\nu$, for large ν, is approximately normal with variance $\simeq 2/(\nu-1)$. Hence, $\ln d^2$ is approximately normal, for large N, with variance $\simeq \frac{1}{p^2} \sum_{j=1}^{p} \frac{2}{N-j-1}$. Thus the approximate percentage points of F_{max} can be determined from

$$F_{p,max}\,(\alpha) = \exp\,[r_k(\alpha)\,\frac{1}{p}\,\{\sum_{1}^{p} 2/(N-j-1)\}^{1/2}]$$

where $r_k(\alpha)$ is the $100\alpha\%$ point of the range r, in independent normal samples of size k. Tabulated values of $r_k(\alpha)$ are available from Pearson and Hartley (1966).

7. Example. The following example is taken from Gnanadesikan and Gupta (1970) who were interested in a ranking and selection procedure

based on generalized variance. They considered 5(=p) - diminesional summaries of speech spectrographic data from a talker identification problem. The data consisted of 7(=N) replicate utterances of 10(=k) words for one particular speaker. Then,

$$F_{p,max} = (720616.4465/1.5411)^{1/5} = 13.6137$$

and $F_{p,max}\ (.01) = 9.0737$

Hence, the hypothesis of equal multidimensional scatter, as measured by SGV, is to be rejected.

8. Unbiasedness of the modified LRT's for H_{01} and H_{02}.

The results in this section show that tests for SGV's possess the property of unbiasedness as do the corresponding tests for covariance matrices. We will consider the modified LRT's obtained by replacing the sample size N_i by the degrees of freedom $n_i = N_i - 1$ in the original LRT, and unbiasedness of these tests for H_{01} and for H_{02} in the case of equidimensional vector variables will be established. The proofs for the covariance matrices were given by Sugiura and Nagao (1968).

8.1. Unbiasedness of modified LRT for $|\Sigma|^{1/p} = \sigma_0^2$.

Theorem 8.1.1. For testing $H_{01}: |\Sigma| = \sigma_0^{2p}$ against the alternative $H_{11}: |\Sigma| \neq \sigma_0^{2p}$ for unknown μ, the modified LRT given by replacing N by $n = N - 1$ in the original LRT is unbiased.

Proof: For given $\sigma_0 > 0$, let $\mathcal{S}_0 = \{\Sigma: |\Sigma| = \sigma_0^{2p}, \Sigma \text{ p.d.}\}$, $\mathcal{S}_1 = \{\Sigma: |\Sigma| \neq \sigma_0^{2p}, \Sigma \text{ p.d.}\}$.

Then

$$\mathcal{S}_0 \cap \mathcal{S}_1 = \phi .$$

To prove Th. 8.1.1, it is then enough to show that for any $\Sigma \in \mathcal{S}_0$, say $\Sigma_0 \in \mathcal{S}_0$ and for any $\Sigma \in \mathcal{S}_1$, say $\Sigma_1 \in \mathcal{S}_1$, (8.1.1) $P(\omega|H_{01},\Sigma_0) - P(\omega|H_{11},\Sigma_1) \le 0$

where ω is the critical region of the modified LRT, i.e.,

$$(8.1.2) \quad \omega: \left[S/S \text{ is p.d. and } \left(\frac{e}{N}\right)^{-Np/2} \left(|S|/\sigma_0^{2p}\right)^{n/2} \exp\left\{-p/2\left(|S|/\sigma_0^{2p}\right)^{1/p}\right\} < c'_\alpha \right]$$

$$(8.1.3) \quad \equiv \omega: \left[S|S \text{ is p.d. and } \left(\frac{e}{N}\right)^{-Np/2} |\Sigma_0^{-1}S|^{(N-1)/2} \exp\left\{-\frac{p}{2}|\Sigma_0^{-1}S|^{1/p}\right\} < c'_\alpha \ \forall \Sigma_0 \in \mathcal{S}_0 \text{, i.e., } |\Sigma_0| = \sigma_0^{2p} \right].$$

This is so because if (8.1.1) is true for any arbitrary $(\Sigma_0,\Sigma_1) \in (\mathcal{S}_0,\mathcal{S}_1)$, then it is also true for any arbitrary $(\Sigma_0,\Sigma^*) \in (\mathcal{S}_0,\mathcal{S}_1)$ or $(\Sigma_0',\Sigma_1) \in (\mathcal{S}_0,\mathcal{S}_1)$ and, hence, this will imply $P(\omega/H_{01}) - P(\omega,H_{11}) \le 0$.
Let $g \in O(p)$, the multiplication group of $p\times p$ orthogonal matrices, be such that $g\,\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}g'$ is a diagonal matrix Γ where $\Sigma_0^{-1/2} = \left(\Sigma_0^{1/2}\right)^{-1}$, $\Sigma_0^{1/2}\Sigma_0^{1/2} = \Sigma_0$. Without any loss of generality we can assume that $\Sigma_0 = I$ and $\Sigma_1 = \Gamma$, the diagonal matrix where diagonal elements are the characteristic roots of $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$. Hence under H_{11}, S has a Wishart distribution with parameters Γ and $n = N - 1$. Now

$$P(\omega|H_{11}) = \int_\omega c_{n,p}\, |S|^{(N-p-2)/2} |\Gamma|^{-(N-1)/2} \exp\{-\tfrac{1}{2}\operatorname{tr}\Gamma^{-1}S\}dS$$

$$= \int_{\omega'} c_{n,p}\, |U|^{(N-p-2)/2} \exp\{-\tfrac{1}{2}\operatorname{tr} U\}dU$$

where

$$c_{n,p}^{-1} = \pi^{p(p-1)/4}\, 2^{np/2} \prod_{i=1}^{p} \Gamma((n-i+1)/2),$$

$$U = \Gamma^{-1/2}\, S\Gamma^{-1/2}$$

and

$$\omega' = \{U:\ U \text{ p.d.},\ \Gamma^{1/2}\, U\Gamma^{1/2} \in \omega\}.$$

Also

$$\left|\frac{\partial U}{\partial S}\right| = |\Gamma|^{-(p+1)/2} .$$

Since $\omega' = \omega$ when H_{01} is true and in the region ω, letting $\theta_i, i = 1,\ldots,p$ be the characteristic root of U,

$$\begin{aligned} f(U) &= |U|^{(N-p-2)/2} \exp(-\tfrac{1}{2} \operatorname{tr} U) = |U|^{(N-p-2)/2} \exp(-\tfrac{1}{2} \Sigma\theta_i) \\ &\le c'_\alpha \ |U|^{-(p+1)/2} \left(\frac{e}{N}\right)^{Np/2} \exp\left[-\frac{p}{2}\left\{\frac{\Sigma\theta_i}{p} - (\ \pi\theta_i)^{1/p}\right\}\right] \\ &\le c'_\alpha \left(\frac{e}{N}\right)^{Np/2} |U|^{-(p+1)/2} = g(U) , \end{aligned}$$

we get

$$\int_{\omega-\omega\cap\omega'} |U|^{(N-p-2)} \exp\{-\tfrac{1}{2} \operatorname{tr} U\} dU$$

exists and

$$\int_{\omega\cap\omega'} g(U)dU < \infty .$$

Further,

$$\int_{\omega-\omega\cap\omega'} f(U)dU \le \int_{\omega-\omega\cap\omega'} g(U)dU$$

and

$$-\int_{\omega^*-\omega\cap\omega'} f(U)dU \le -\int_{\omega'-\omega\cap\omega^*} g(U)dU .$$

Thus, we get

$$\begin{aligned} &P(\omega|H_{01},\Sigma_0) - P(\omega|H_{11},\Sigma_1) \\ &= c_{n,p}\left\{\int_{\omega-\omega\cap\omega^*} - \int_{\omega^*-\omega\cap\omega^*} f(U)dU\right\} \\ &\le c_{n,p}\ c'_\alpha\left(\frac{e}{N}\right)^{Np/2}\left\{\int_{\omega-\omega\cap\omega^*} - \int_{\omega^*-\omega\cap\omega^*}\right\}|U|^{-(p+1)/2}dU \end{aligned}$$

$$= c_{n,p} c'_\alpha \left(\frac{e}{N}\right)^{Np/2} \left\{\int_\omega - \int_{\omega^*}\right\} |U|^{-(p+1)/2}$$

The last inequality is due to the fact that $|U|^{-(p+1)/2}$ is the invariant measure in the space of U under the full linear group $G_\ell(p)$ transforming $U \to gUg'$, $g \in G_\ell(p)$, that is,

$$\int_{\omega'} |U|^{-(p+1)/2} dU = \int_\omega |U|^{-(p+1)/2} dU \ . \qquad \text{Q.E.D.}$$

Precisely this property was exploited by Sugiura and Nagao to prove the unbiasedness of the modified LRT's for $\Sigma = \Sigma_0$ and $\Sigma_1 = \Sigma_2$.

Theorem 8.1.1 can be generalized to the k-sample case. Let $x_{ji}: p\times 1$, $i=1,\ldots,N_j$ $(N_j>p)$ be a random sample from $N_p(\mu,\Sigma_j)$, $j=1,2,\ldots,k$. Let S_j be the sample sums of products matrix and $n_j = N_j-1$. Using the same argument as in the proof of Theorem 6.1.1 we have the following theorem.

<u>Theorem 8.1.2</u>. For testing the hypothesis $H'_{01}: |\Sigma_j|^{1/p} = \sigma^2_{0j}$ $(j=1,2,\ldots,k)$ against the alternatives $H'_{11}: |\Sigma_i|^{1/p} \neq \sigma^2_{0i}$ for some i, where the mean μ_j is unspecified and $\sigma^2_{0j} > 0$ is given, the modified LRT having the critical region

$$\omega' : \Bigg[(S_1,\ldots,S_k)/S_j \ \text{ is p.d. } (j=1,\ldots k)$$

and

$$\prod_{j=1}^{k} \left(\frac{e}{N_j}\right)^{-N_j p/2} \left(|S_j|/\sigma^{2p}_{0j}\right)^{n_j/2} \exp\left\{-\frac{p}{2}\left(\frac{|S_j|}{\sigma^{2p}_{0j}}\right)^{1/p}\right\} < c''_\alpha \Bigg]$$

is unbiased.

Consider further the problem of testing $H''_{01}: |\Sigma|^{1/p} = \sigma_0^2$ against $H''_{11}: |\Sigma|^{1/p} > \sigma_1^2$ where $0 < \sigma_0 < \sigma_1$. As a modification of the LRT in Section 3.1 consider the test ϕ_0,

$$\phi_0 \text{: Reject } H''_{01} \text{ iff } |S|/\sigma_0^{2p} > c_\alpha$$

<u>Theorem .1.3</u>. For testing H''_{01} against H''_{11} , the test ϕ_0 is the uniformly most powerful invariant. Also ϕ_0 is a maximum test and is most stringent.

<u>Proof</u>. See proof of Proposition 1 of Eaton (1967).

<u>8.2. Unbiasedness of the modified LRT for</u> $|\Sigma_1| = |\Sigma_2|$. Let $x_i \sim N_p(\mu_1,\Sigma_1)$, $i=1,\ldots,N_1$ and $\underset{\sim}{y}_j$ $N_p(\mu_2,\Sigma_2)$, $j=1,\ldots,N_2$ denote two independent random samples from two independent multivariate normal populations. For testing $H_{02}: |\Sigma_1|^{1/p} = |\Sigma_2|^{1/p}$ (or equivalently $|\Sigma_1| = |\Sigma_2|$) against $H_{12}: |\Sigma_1|^{1/p} \neq |\Sigma_2|^{1/p}$, the critical region of LRT is given by, from (3.2.1), as

$$\text{(8.2.1)} \quad \omega = \left\{ (S_1,S_2): \frac{\left(s_1^2\right)^{N_1p/2} \left(s_2^2\right)^{N_2p/2}}{\left(\dfrac{s_1^2+s_2^2}{N_1+N_2}\right)^{(N_1+N_2)p/2}} < c'_\alpha \text{ , a constant} \right\}$$

where $s_i^{2p} = |S_i|$, S_i , $i=1,2$ are the sample sums of products matrices for X and Y respectively.

<u>Theorem 8.2.1</u>. The modified LRT for H_{02} $|\Sigma_1| = |\Sigma_2|$ against H_{12}: $|\Sigma_1| \neq |\Sigma_2|$ obtained by replacing N_i by $n_i = N_i-1$, $i=1,2$ in (8.2.1) is unbiased.

Proof. As in Section 8.1, let $S_o = \{\tilde{\Sigma}:(\Sigma_1,\Sigma_2):|\Sigma_1| = |\Sigma_2|;\Sigma_1,\Sigma_2 \text{p.d.}\}$ and $S_1 = \{\tilde{\Sigma}:(\Sigma_1,\Sigma_2):|\Sigma_1| \neq |\Sigma_2|;\Sigma_1,\Sigma_2 \text{p.d.}\}$. Then, $S_o \cap S_1 = \phi$. To prove Th. 8.2.1 it suffices to show that for any $\tilde{\Sigma}\epsilon S_o$, say $\tilde{\Sigma}_o$, and for any $\tilde{\Sigma}\epsilon S_1$, say $\tilde{\Sigma}_1$,

$$p(\omega|H_{o2},\tilde{\Sigma}_o) - p(\omega|H_{12},\tilde{\Sigma}_1) \leq 0 .$$

Consider any $\tilde{\Sigma} = (\Sigma_1,\Sigma_2)$.

Without loss of generality, we can take $\Sigma_2 = I$ and $\Sigma_1 = \theta$, the diagonal matrix with diagonal elements $\theta_1,\ldots,\theta_p$.

$$P(\omega|H_{12}) = c_{n_1,p}\, c_{n_2,p} \int_{(S_1,S_2)\in\omega} |S_1|^{(n_1-p-1)/2} \; |S_2|^{(n_2-p-1)/2}$$

$$|\theta|^{-n_1/2} \exp\left\{-\frac{1}{2}\operatorname{tr}(\theta^{-1}S_1+S_2)\right\} dS_1 dS_2$$

$$= c_{n_1,p}\, c_{n_2,p} \int_{(I,u_2)\in\omega} |U_1|^{(n-p-1)/2} \; |U_2|^{(n_2-p-1)/2}$$

$$|\theta|^{-n_1/2} \exp\left\{-\frac{1}{2}\operatorname{tr}(\theta^{-1}+U_2)U_1\right\} dU_1 dU_2$$

$$= b \int_{(I,U_2)\in\omega} |U_2|^{(n_2-p-1)/2} \; |\theta|^{-n_1/2} \; |(\theta^{-1}+U_2)|^{-n/2} dU_2$$

where $S_1 = U_1$, $S_2 = U_1^{1/2}U_2U_1^{1/2}$ with $U_1^{1/2}$ a symmetric matrix such that $U_1 = U_1^{1/2}U_1^{1/2}$ and $b = c_{n_1,p}\, c_{n_2,p}/c_{n,p}$, $n = n_1+n_2$. The Jacobian of the transformation $(S_1,S_2) \to (U_1,U_2)$ is given by

$$\left| \frac{\partial(S_1,S_2)}{\partial(U_1,U_2)} \right| = |U_1|^{(p+1)/2}$$

Put $V = \theta^{1/2}\, V_2\, \theta^{1/2}$. Let ω^* be the set of all $p \times p$ positive definite matrices V such that $(I, \theta^{-1/2}\, V\, \theta^{-1/2}) \in \omega$, and $\tilde{\omega}$ be the set of all $p \times p$ positive definite symmetric matrices V such that $(I,V) \in \omega$. Then,

$$P(\omega|H_{02}) - P(\omega|H_{12}) = b\left\{\int_{\bar{\omega}} - \int_{\omega^*}\right\} |V|^{(n_2-p-1)/2} |(I+V)|^{-n/2} dV$$

$$= b\left\{\int_{\bar{\omega}-\bar{\omega}\cap\omega^*} - \int_{\omega^*-\bar{\omega}\cap\omega^*}\right\} |V|^{n_2/2} |(I+V)|^{-n/2}\, |V|^{-(p+1)/2} dV .$$

Now consider the following.

<u>Lemma</u>. If S_1 and S_2 are two positive definite matrices of the same dimension, $p \times p$, then

$$\{|S_1|^{1/p} + |S_2|^{1/p}\}^p < |S_1+S_2| .$$

<u>Proof</u>.[1] Since S_1 and S_2 are two p.d. matrices, there exists a nonsingular matrix M such that $MS_1M' = I$ and $MS_2M' = \Lambda =$ diagonal $(\delta_1,\ldots,\delta_p)$ where $\delta_i > 0$, $i=1,\ldots,p$ are the characteristic roots of $S_2S_1^{-1}$. To prove the lemma, it is enough to show

[1] **Professor Olkin has pointed out that this can be proven alternatively through the concavity of $|S|^{1/p}$.**

(3) $$\{|MS_1M'|^{1/p}+|MS_2M'|^{1/p}\}^p < |MS_1M'+MS_2M'|$$

$$\text{i.e.} \quad \{|I|^{1/p}+|\Lambda|^{1/p}\}^p < \{|I+\Lambda|\}$$

$$\text{i.e.} \quad \left\{1+\prod_{i=1}^{p} \delta_i^{1/p}\right\} < \prod_{i=1}^{p} (1+\delta_i)^{1/p}$$

which is true by Holder's Inequality. Hence the lemma.

$$P(\omega|H_{02}) - P(\omega|H_{12}) \leq bc'_\alpha\left\{\int_{\bar{\omega}-\bar{\omega}\cap\omega^*} - \int_{\omega^*\bar{\omega}\cap\omega^*}\right\} |V|^{-(p+1)/2}dV$$

$$= bc'\left\{\int_{\bar{\omega}} - \int_{\omega^*}\right\}|V|^{-(p+1)/2}dV = 0 \ ,$$

since

$$\int_{\bar{\omega}}|V|^{(n_2-p-1)/2}|(I+V)|^{-n/2}dV < \infty$$

and for any subset ω' of $\bar{\omega}$

$$\int_{\omega'}|V|^{(n_2-p-1)/2}|(I+V)|^{-n/2}dV \leq \int_{\omega'}|V|^{(n_2-p-1)/2}\left[\left(1+|V|^{1/p}\right)^p\right]^{-n/2}dV$$

(by the Lemma)

$$\leq c'_\alpha \int_{\omega'}|V|^{-(p+1)/2}dV < \infty \ . \quad \text{Q.E.D.}$$

9. Remarks. As with any multidimensional measure, the SGV cannot be expected to be the unique measure best for all situations of multidimensional scatter. However, if we are interested in 'overall' scatter and where magnitude of individual variances separately are not of great concern, the SGV can be expected to perform adequately. Wilks (1967) gives an expository account of GV as a measure of multidimensional scatter from geometrical standpoint.

This paper suggests several topics for future research. Admissibility of the LRTs may be studied. Also, alternative test procedures to LRTs, e.g., by the Union-Intersection method will be interesting. Sequential and nonparametric test procedures for SGVs may provide further insights into the problem. The case of one sided alternatives and singular dispersion matrices seem to be of great practical importance and interest. Finally, one can explore situations where SGVs may not be adequate and for such situations provide adequate measures and tests for multidimensional scatter.

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The concept of Standardized Generalized Variances (SGV's) is introduced. Several new problems of multivariate statistical inference are formulated on the basis of these SGV's. It is shown that in addition to providing several new statistical tests, many existing problems of multivariate tests of significance can be regarded as special cases of these formulations and can also be extended to their full generalities. Considering multivariate normal populations with general covariance matrices, likelihood ratio tests are derived for SGV's. The criteria turn out to be elegant multivariate analogues to those for tests of variances in the univariate cases. A multivariate F_{max} criterion is also presented as an alternative shortcut method for the case of k populations. The null and nonnull distributions of the test criteria are deduced in computable forms [see, e.g., Mathai et al. (1979)] in terms of special functions, e.g., Pincherle's H-function, Meijer's G-function, etc., by exploiting the theory of calculus of residues. Some large sample approximations to these distributions are also proposed. The property of unbiasedness for the modified likelihood ratio tests is established for some of the above test criteria. Finally, applications of the above tests to a wide spectrum of applied research are illustrated by examples taken from the existing literature, e.g., Gnanadesikan (1977), Gnanadesikan and Gupta (1970), etc.

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